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$H\infty$ Control for Impulsive Disturbances:
A State-Space Solution

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H_∞ CONTROL FOR IMPULSIVE DISTURBANCES: A STATE-SPACE SOLUTION *

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Abstract

In this paper we formulate and study an interesting (sub)optimal H_∞ control problem related to the attenuation of impulsive disturbances to a class of linear systems. Among the motivating factors is the need to study control problems related to mechanical systems subject to impulsive forces, e.g. active control of the suspension system of a vehicle, accurate pointing of guns, stabilization of an antenna on the space station subject to impact from space debris, or active damping of vibrations of flexible structures caused by impact forces [1, 2]. A reasonable control objective in all these problems is to design a stabilizing controller to minimize the induced operator norm from the impulsive disturbances to the controlled outputs. We derive necessary and sufficient conditions for the existence of a (sub)optimal controller, and give a procedure to compute such a controller when one exists.

1   Introduction

An important paradigm in control synthesis is the H_∞ control problem, introduced by Zames [3]. In this formulation disturbances are assumed to belong to a ball in a certain function space, and control inputs are computed in order to minimize a quadratic cost function assuming that the disturbances are worst possible. Various theories of H_∞ control problems have been developed by many researches [4, 5, 6, 7, 8, 9] and procedures for solving them have been given via frequency domain methods and more recently, using state-space methods. In this paper, we consider a (sub)optimal H_∞ control problem for the attenuation of dynamical effects due to an impulsive disturbance. Motivation for this arises from the need to study control

problems related to mechanical systems subject to impulsive forces, e.g. active control of the suspension system of a vehicle, accurate pointing of guns, stabilization of an antenna on the space station subject to impact from space debris, or active damping of vibrations of flexible structures caused by impact forces. A reasonable control objective in all these problems is to design a stabilizing controller to minimize the induced operator norm from the impulsive disturbance to the controlled output. Our approach is based on the state-space method[10]. Impulsive disturbance is denoted by v_0 and is assumed to live in the l_2 space. The controlled output is denoted by z. Our goals are (1) to give necessary and sufficient conditions for the existence of a stabilizing controller such that $||z||_{L_2} \leq \gamma||v_0||_{l_2}$ for all $v_0 \neq 0$ and a given γ, and (2) to give a procedure to compute such a controller when one exists.

2 Formulation of Control Problem

In this section, the control problem is formulated with the specification of system dynamics, performance index, and general assumptions. Let l_2 denote the Hilbert space of square-summable complex-valued sequences $\{x_k, k \geq 0\}$ with the norm defined as

$$||x||_{l_2} := (\sum_{k=0}^{\infty} |x_k|^2)^{1/2}. \tag{2.1}$$

Let L_2 denotes the Hilbert space of all complex-valued Lebesque measurable functions $x(t)$ defined on the $[0, \infty)$ with the property that $|x|^2$ is Lebesque integrable. The norm

$$||x||_{L_2} := (\int_0^{\infty} ||x(t)||^2 dt)^{1/2}. \tag{2.2}$$

We consider a finite-dimensional linear system

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} v_0 \\ u \end{bmatrix} \tag{2.3}$$

where $z(t), y(t)$ are controlled and measured output vector, respectively. They are piecewise-continuous signals. There are two kinds of input signals, u the control vector which is assumed to be piecewise-continuous, and v_0 the impulsive disturbance vector which is assumed to be in the l_2 space. $L_{ij}, i, j = 1, 2$ are the linear operators mapping from $\{v_0, u\}$ to $\{z, y\}$. This is a hybrid system since it contains both continuous time and discrete time components. We assume that the system (2.3) admits a state space realization of the form:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \sum_{i=0}^{\infty} B_2 v_0(i)\delta(t - iT_i) + B_1 u(t), \\
z(t) &= C_1 x(t) + D_1 u(t), \\
y(t) &= C_2 x(t),
\end{aligned} \tag{2.4}$$

where $x(t)$ are $n \times 1$ state variables. $A, B_1, B_2, C_1, C_2, D_1$ are constant matrices with proper dimensions. T_i is the interval between the $(i-1)^{th}$ and i^{th} impulses. In the rest of this paper, we assume that this time interval is constant, i.e., $T_i = T$, $i = 0, 1, 2, \cdots\cdots$. The $\delta(t)$ is the standard Dirac Delta distribution. It is easy to see that the effect of impulsive disturbances is to cause possible jumps of state variables. For convenience, the state space equations (2.4) can be put into the following form also:

$$
\begin{array}{rcl}
\dot{x}(t) & = & Ax(t) + B_1 u(t), \quad t \neq iT, \\
x(t^+) & = & x(t) + B_2 v_0(i), \quad t = iT, \quad i = 0, 1, 2, \cdots\cdots, \\
z(t) & = & C_1 x(t) + D_1 u(t) \\
y(t) & = & C_2 x(t)
\end{array}
\tag{2.5}
$$

where $x(t^+)$ denote the values of state variables after jump. The state variables $x(t)$ are right continuous and may be left discontinuous due to the possible jump.

The control problem here is to design a stablilizing controller to attenuate effects of the impulsive disturbance v_0 on the controlled output z. This problem can be studied in the H_∞ control framework [3, 4, 5]. The solution for a special case of this problem by frequency-domain methods is given in [11]. Let us define K as the set of all causal, finite-dimension linear stabilizing controllers.

We will now introduce a minimax performance measure which is motivated by [10]. For $k \in K$ define a performance measure,

$$
J(k) = \sup_{v_0 \in l_2, v_0 \neq 0} \left(\frac{||z||_{L_2}^2}{||v_0||_{l_2}^2} \right).
\tag{2.6}
$$

$J(k)$ can also be viewed as the induced norm of the linear operator from $l_2 \to L_2$. The control objective is to find a controller $k \in K$ to minimize the worst case performance measure $J(k)$. Specifically, we solve the following (sub)optimal problem. Given a real $\gamma > 0$, try to find a $k \in K$ such that following inequality holds,

$$
J(k) < \gamma^2
\tag{2.7}
$$

or equivalently,

$$
||z||_{L_2}^2 - \gamma^2 ||v_0||_{l_2}^2 < 0,
\tag{2.8}
$$

for all possible $v_0 \in l_2$, under the constraints of system equations (2.5). If such a controller k exists, we call it a $\gamma-$level disturbance attenuation controller.

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3 Main Results

We start with a standard H_∞ control problem treated in [5, 6] where we consider a continuous LTI system. It is well known that a state-space solution of H_∞ control is closely related to the Riccati equations. If full state-feedback is available, a controller exists iff the unique solution of the associated algebraic Riccati equation is positive definite. In addition, a formula for the state-feedback gain matrix was given in terms of the solution of the Riccati equation. Similar results can be obtained for discrete-time systems [8] and time-varying systems[12].

It should be noted that the control problem defined in (2.8) is different from the standard H_∞ problems treated in [5, 6] or [8, 12]. The system (2.5) is a hybrid system which contains both continuous and discrete components, the formulas obtained in [5, 6, 8, 12] can not be applied to this problem, but the essential ideas can be carried over to analyze the control problem (2.8). For state-feedback control, instead of one algebric Riccati equation involved in problems [5, 6], we will have two coupled Riccati equations given by

$$\dot{K}(t) = -A'K(t) - K(t)A + K(t)B_1B_1'K(t) - C_1'C_1, \quad t \neq iT \quad (3.1)$$

$$K(iT^+) = K(iT) + (B_2'K(iT))'(\gamma^2 I - B_2'K(iT)B_2)^{-1}B_2'K(iT) \quad (3.2)$$

where $\gamma > 0$ is a given real number and $T > 0$ is the time interval between each jump. The $n \times n$ matrix-valued function $K(t)$ is right continuous and may be left discontinuous. We use $K(iT)$ to represent the value of $K(t)$ just before the ith jump, i.e., $K(iT) := \lim_{\epsilon > 0, \epsilon \to 0} K(iT - \epsilon)$ assuming that the limit exists.

We state the our main results as Theorems 3.1-3.3.

We assume that all state variables are measured. We consider the infinite horizon problem with zero initial state. First the following standard assumptions are made.

i) (A, B_1) is controllable, (C_1, A) is observable.
ii) $D_1'[C_1 \; D_1] = [0 \; I]$.

Assumption i) is standard in the quadratic regulator of linear system. It can be relaxed by the assumption that (A, B_1) is stabilizable and (C_1, A) is detectable. Assumption ii) is made here just for the sake of simplicity. Relaxing this assumption will only complicate the formulas, but an analysis can be carried out along lines similar to what appears below.

Theorem 3.1 *Consider the hybrid system described by (2.5). Let $\gamma > 0$ be given. Suppose that the assumptions i) and ii) hold. Then there exists a stabilizing controller $k \in K$ such that $J(k) < \gamma^2$ if there exists a unique stabilizing positive definite*

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periodic solution $P(t)$ of the coupled Riccati equations (3.1)-(3.2). Moreover, if this condition is satisfied, one such stabilizing state-feedback controller is given by

$$u(t) = -B_1' P(t) x(t), \quad \forall t > 0. \tag{3.3}$$

$$\square$$

Remark 3.1 *As $\gamma \to \infty$, the equations (3.1)-(3.2) degenerate into a continuous-time Riccati equation,*

$$\dot{K}(t) = -A'K(t) - K(t)A + K(t)B_1 B_1' K(t) - C_1' C_1, \tag{3.4}$$

This Riccati equation will yield a unique postive definite constant matrix solution under the assumption i). The unique solution internally stabilizes the associated closed-loop system by the standard LQG theory. Thus, the H_∞ problem degenerates into a LQG problem as $\gamma \to \infty$.

A very useful tool to analyze the Riccati equations is the Hamiltonian theory, this solution of a Riccati equation can be obtained by solving a suitable set of linear differential equations. The Hamiltonian matrix associated with continuous time Riccati equation (3.1) is given by

$$H = \begin{bmatrix} A & -B_1 B_1' \\ -C_1' C_1 & -A' \end{bmatrix}. \tag{3.5}$$

As usual let us consider a 2n order differential equation,

$$\dot{X}(t) = HX(t), \quad t \neq iT. \tag{3.6}$$

The state transition matrix associated with H is

$$X(t) = \Phi_H(t, \tau) X(\tau), \quad t \geq \tau, \quad and \quad t, \tau \neq iT, \tag{3.7}$$

where $\Phi_H(t, \tau)$ has the following properties,

$$\frac{\partial \Phi_H(t, \tau)}{\partial t} = H\Phi_H(t, \tau), \quad \Phi_H(\tau, \tau) = I. \tag{3.8}$$

The symplectic matrix associated with difference Riccati equation (3.2) is given by

$$F = \begin{bmatrix} I & -\gamma^{-2} B_2 B_2' \\ 0 & I \end{bmatrix}. \tag{3.9}$$

Let us define a 2n order difference(jump) equation as follows,

$$X(t^+) = FX(t), \quad t = iT. \tag{3.10}$$

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The combined equations (3.6) and (3.10) is a hybrid system. For $\tau = iT, i = 0, 1, \cdots, \cdots$, and $\tau \leq t \leq (i+1)T$, we have,

$$
\begin{aligned}
X(t) &= \Phi_H(t, \tau) F X(\tau), \\
&= \Phi(t, \tau) X(\tau).
\end{aligned} \tag{3.11}
$$

We will show later that this state transition matrix $\Phi(t, \tau)$ of the hybrid system displays same properties familiar with a periodic system.

Now, let $P(t)$ be a periodic solution of equations (3.1)-(3.2). We will study the associated closed-loop system,

$$
\dot{x}(t) = [A - B_1 B_1' P(t)] x(t). \tag{3.12}
$$

Definition 3.1 *A periodic solution $P(t)$ of (3.1)-(3.2) is called a stabilizing solution if the closed-loop system (3.12) is asymptotically stable.*

Let us consider the following hybrid system first,

$$
\begin{aligned}
\dot{x}(t) &= [A - B_1 B_1' P(t)] x(t), \\
&= A_c(t), \quad t \neq iT, \tag{3.13} \\
x(t^+) &= [I - \gamma^{-2} B_2 B_2' P(t)] x(t), \\
&= F_c(t), \quad t = iT. \tag{3.14}
\end{aligned}
$$

It is easy to see that this hybrid system is actually the closed-loop system for an impulsive disturbance generated by state-feedback. Thus, the asymptotically stability of this hybrid system provides a necessary condition for asymptotically stabilizing of closed-loop system (3.12).

Theorem 3.2 *Let $\gamma > 0$ be given. Suppose that the assumptions i) and ii) hold. Then a necessary and sufficient condition for the existence of a unique positive definite periodic solution $P(t)$ to equations (3.1) - (3.2) such that the hybrid system (3.13)-(3.14) is asymptotically stable is that no eigenvalue of $\Phi(T, 0)$ lies on the unit circle.* □

The next theorem gives necessary and sufficient conditions that the unique positive definite periodic solution stabilizes the associated closed-loop system.

Theorem 3.3 *Suppose that the assumptions i) and ii) hold. Let $P(t)$ be the unique positive definite periodic solution of (3.1) and (3.2). Then $P(t)$ is a stabilizing solution iff the following inequality holds,*

$$
W_c(T) - \Phi_{A_c}'(T, 0) Q_{tmp}(T) \Phi_{A_c}(T, 0) > 0 \tag{3.15}
$$

where,

$$W_c(T) = \int_0^T \Phi_{A_c}(\tau,0)'H(\tau)'H(\tau)\Phi_{A_c}(\tau,0)d\tau \qquad (3.16)$$

and $\Phi_{A_c}(t,\tau)$ is state transition matrix of $A_c(t)$. The matrices $A_c(t), H(t)$ and $Q_{tmp}(T)$ are defined by

$$
\begin{aligned}
A_c(t) &= A - B_1 B_1' P(t), \\
H(t)'H(t) &= C_1' C_1 + P(t) B_1 B_1' P(t), \\
Q_{tmp}(T) &= (B_2' P(T))'(\gamma^2 I - B_2' P(T) B_2)^{-1} B_2' P(T). \qquad (3.17)
\end{aligned}
$$

$\square$

4 Background and Technical Lemmas

In order to prove Theorems 3.1-3.3, we need to develop theory and technical machinery to analyze the coupled Riccati equations (3.1) and (3.2). Specifically, 1) we will give conditions for existence of periodic solutions of equations (3.1) and (3.2), 2) If such conditions are satisfied, we will parametrize all stabilizing periodic solutions. The technical machinery is built here based on the analysis of the standard periodic systems [13, 14, 15]. We will show that equations (3.1) and (3.2) display same properties familar with a standard periodic system, the main difference being that we need to take care of the jumps in the state which actually result in the periodic solutions. For the sake of simplicity, we are not going to give all proofs to the lemmas and theorems developed in this section. The complete proofs can be found in [16] and similar proofs for a standard periodic system can be found in [13].

Let $\Phi(t,\tau)$ defined in (3.11) be partitioned into four $n \times n$ matrices as follows,

$$\Phi(t,\tau) = \begin{bmatrix} \Phi_{11}(t,\tau) & \Phi_{12}(t,\tau) \\ \Phi_{21}(t,\tau) & \Phi_{22}(t,\tau) \end{bmatrix}. \qquad (4.1)$$

It is easy to show that $\Phi(t,\tau)$ has the property for all t and τ,

$$\Phi(t+T,\tau+T) = \Phi(t,\tau). \qquad (4.2)$$

Lemma 4.1 *The eigenvalues of $\Phi(t+T,t)$ are independent of t.*

Proof: *Without loss generality, we assume $t \in (0,T)$. By definition,*

$$
\begin{aligned}
\Phi(t+T,t) &= \Phi_H(t+T,T^+)F\Phi_H(T,t) \\
&= \Phi_H(t,0^+)F\Phi_H(T,t) \\
&= \Phi_H^{-1}(T,t)\Phi_H(T,0^+)F\Phi_H(T,t) \\
&= \Phi_H^{-1}(T,t)\Phi(T,0)\Phi_H(T,t).
\end{aligned}
$$

Thus, the eigenvalues of $\Phi(t+T,t)$ are independent of t. $\square$

The hybrid state transition matrix $\Phi(t+T, t)$ displays same properties familiar with periodic systems. Hence we may expect that there exists a periodic solution of equations (3.1)-(3.2). For simplicity, we will assume that the matrix $\Phi(T, 0)$ has distinct eigenvalues. Let Λ be the corresponding diagonal Jordan form such that

$$S^{-1}(0)\Phi(T, 0)S(0) = \Lambda, \tag{4.3}$$

where the matrix $S(0)$ is composed of eigenvectors of $\Phi(T, 0)$. It should be noted that the derivation of results can be applied to the case of multiple eigenvalues with minor modifications. Now partition $S(0)$ into four $n \times n$ matrices

$$S(0) = \begin{bmatrix} Y(0) & V(0) \\ X(0) & U(0) \end{bmatrix} \tag{4.4}$$

and partition Λ similarly. Then equation (4.3) can be written as,

$$\begin{bmatrix} \Phi_{11}(T, 0) & \Phi_{12}(T, 0) \\ \Phi_{21}(T, 0) & \Phi_{22}(T, 0) \end{bmatrix} \begin{bmatrix} Y(0) & V(0) \\ X(0) & U(0) \end{bmatrix} = \begin{bmatrix} Y(0) & V(0) \\ X(0) & U(0) \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{4.5}$$

where Λ_1 is an $n \times n$ diagonal matrix,

$$\Lambda_1 = diag\{\lambda_1, \cdots\cdots, \lambda_n\} \tag{4.6}$$

and $\{\lambda_1, \cdots\cdots, \lambda_n\}$ are n of the 2n eigenvalues of $\Phi(T, 0)$. Let us define a $2n \times 2n$ matrix $S(t)$ for $t \geq 0$ by

$$S(t) = \Phi(t, 0)S(0). \tag{4.7}$$

Again, partition $S(t)$ as follows,

$$S(t) = \begin{bmatrix} Y(t) & V(t) \\ X(t) & U(t) \end{bmatrix} \tag{4.8}$$

Lemma 4.2 *The equation* $\Phi(t + T, t)S(t) = S(t)\Lambda$ *holds for all* $t \geq 0$. $\qquad\square$

It is easy to show that the following equation also holds,

$$\begin{aligned} \frac{dS(t)}{dt} &= HS(t), \quad t \neq iT, \\ S(t^+) &= FS(t), \quad t = iT, i = 0, 1, 2, \cdots\cdots, \end{aligned} \tag{4.9}$$

Theorem 4.1 *Let* λ *be an eigenvalue of* $\Phi(t, \tau)$ *and* $\begin{bmatrix} y \\ x \end{bmatrix}$ *be the corresponding eigenvector. Then* λ^{-1} *is also an eigenvalue of* $\Phi(t, \tau)$ *and* $\begin{bmatrix} -x \\ y \end{bmatrix}$ *is the eigenvector of* $\Phi(t, \tau)'$ *associated with* λ^{-1}. $\qquad\square$

4.1 Parameterization of periodic solution with jumps

The following theorem gives a parameterization of all periodic solutions of equations (3.1)-(3.2) in terms of $X(t)$ and $Y(t)$ defined in equation (4.8).

Theorem 4.2 *Suppose $Y(t)$ defined in equation (4.8) is non-singular for all $t \in [0,T]$. Then $P(t)$ given by*

$$P(t) = X(t)Y(t)^{-1}, \quad t \geq 0 \tag{4.10}$$

is a periodic solution of equation (3.1) and (3.2).

Proof:

$$
\begin{aligned}
S(t+T) &= \Phi(t+T,0)S(0) \\
&= \Phi(t+T,t)\Phi(t,0)S(0) \\
&= S(t)\Lambda.
\end{aligned}
$$

Therefore,

$$
\begin{bmatrix} Y(t+T) \\ X(t+T) \end{bmatrix} = \begin{bmatrix} Y(t) \\ X(t) \end{bmatrix} \Lambda_1, \quad t \geq 0. \tag{4.11}
$$

$$
\begin{aligned}
\dot{P}(t) &= \dot{X}(t)Y(t)^{-1} - X(t)Y(t)^{-1}\dot{Y}(t)Y(t)^{-1} \\
&= [-C_1'C_1 Y(t) - A'X(t)]Y(t)^{-1} - X(t)Y(t)^{-1} \\
&\quad [AY(t) - B_1 B_1' X(t)]Y(t)^{-1} \\
&= -A'P(t) - P(t)A + P(t)B_1 B_1' P(t) - C_1'C_1 \\
P(0^+) &= X(0^+)Y(0^+)^{-1} \\
&= X(0)[Y(0) - \gamma^{-2}B_2 B_2' X(0)]^{-1} \\
&= P(0)[I - \gamma^{-2}B_2 B_2' P(0)]^{-1} \tag{4.12}
\end{aligned}
$$

It is easy to show that $P(0^+)$ satisfies equation (3.2). Thus $P(t)$ in equation (4.10) satisfies equations (3.1) and (3.2). It is periodic since from (4.11),

$$P(t+T) = X(t+T)Y(t+T)^{-1} = X(t)\Lambda_1[Y(t)\Lambda_1]^{-1} = P(t). \tag{4.13}$$

$\square$

Remark 4.1 *It is obvious that periodic solution is generated by jump terms. If $\gamma \to \infty$, the jump terms vanish and $P(t)$ will degenerate into a constant matrix solution.*

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4.2 Analysis of periodic solutions with jumps

The following Lemmas(4.3)-(4.6) characterize the properties of a solution $P(t)$ that depend on the choice of n eigenvalues $\lambda_1, \cdots\cdots \lambda_n$ for Λ_1. Let

$$\Omega(t) = Y(t)^* X(t), \tag{4.14}$$

$$\hat{\Omega}(t) = Y(t)' X(t), \tag{4.15}$$

where $*$ is used to denote the complex conjugate transpose. Suppose $|Y(t)| \neq 0$ for all $t \in [0, T]$, $P(t)$ can be rewritten as,

$$\begin{aligned}
P(t) &= X(t)Y(t)^{-1} \\
&= (Y(t)^{-1})^* \Omega(t) Y(t)^{-1} \\
&= (Y(t)^{-1})' \hat{\Omega}(t) Y(t)^{-1}.
\end{aligned} \tag{4.16}$$

Lemma 4.3 *If $\lambda_i^* \neq \lambda_j^{-1}$ for all $i, j, 1 \leq i, j \leq n$, then $\Omega(t)$ is Hermitian and real.*
□

Lemma 4.4 *If $|\lambda_i| < 1$ for all $i = 1, 2, \cdots\cdots n$, then $\Omega(t)$ is positive definite.* □

Lemma 4.5 *Assume that $|Y(t)| \neq 0$ for all $t \in [0, T]$. If $\lambda_i^* \neq \lambda_j^{-1}$ and $\lambda_i \neq \lambda_j^{-1}$ for all $i, j, 1 \leq i, j \leq n$, then $P(t)$ defined by equation (4.10) is real.* □

Theorem 4.3 *Assume that for all eigenvalues of $\Phi(T, 0)$, $|\lambda_i| \neq 1$, $i = 1, 2 \cdots\cdots 2n$ and that $|Y(t)| \neq 0$ for all $t \in [0, T]$. If n eigenvalues in Λ_1 are chosen such that $|\lambda_i| < 1, i = 1, 2 \cdots\cdots n$, then a periodic solution $P(t)$ given by equation (4.10) is real, symmetric and positive definite.*

Proof: *The assumption $|\lambda_i| < 1$ for all $i = 1, 2 \cdots\cdots n$ guarantees that $\lambda_i^* \neq \lambda_j^{-1}$ and $\lambda_i \neq \lambda_j^{-1}$ for all $i, j, 1 \leq i, j \leq n$. Therefore $P(t)$ is real and symmetric by Lemmas (4.3) and (4.5) respectively. That $P(t)$ is positive definite follows from Lemma (4.4) and equation (4.16).* □

4.3 The Periodic Lyapunov Equation

Our approach for the analysis of the closed-loop system (3.12) is based on the Lyapunov method. As is well-known, the Lyapunov equation plays an important role in the analysis of Riccati equations and the associated closed-loop systems. Here we extend some useful results on the periodic Lyapunov equation to our particular problem, namely, to the periodic Lyapunov equation with jumps.

Let $P(t)$ be a solution of the coupled Riccati equations (3.1)-(3.2),

$$\dot{P}(t) = -A'P(t) - P(t)A + P(t)B_1 B_1' P(t) - C_1' C_1, \quad t \neq iT \tag{4.17}$$

$$P(iT^+) = P(iT) + (B_2' P(iT))'(\gamma^2 I - B_2' P(iT)B_2)^{-1} B_2' P(iT). \tag{4.18}$$

It can be rewritten as a Lyapunov-type equation,

$$\dot{P}(t) \quad = \quad -A_c(t)'P(t) - P(t)A_c(t) - H(t)'H(t), \quad t \neq iT \qquad (4.19)$$

$$P(iT^+) \quad = \quad P(iT) + Q_{tmp}(iT). \qquad (4.20)$$

where $A_c(t), H(t)$ and $Q_{tmp}(iT)$ are given in (3.17). Therefore, we can obtain the following results from [17, 18]. If $P(\cdot)$ is a symmetric periodic solution of the Riccati equation (4.17)-(4.18), then $P(\cdot)$ is also a solution of the Lyapunov equation (4.19)-(4.20). The structural properties of the pair $(H(t), A_c(t))$ can be related to the ones of the pair (C_1, A) by means of the following Lemma [18, 19].

Lemma 4.6 *The pair (C_1, A) is observable iff, for any $T-$periodic matrix $P(\cdot)$, the pair $(H(t), A_c(t))$ is observable.* $\qquad\qquad\Box$

The solution of equations (4.19)-(4.20) is given by the celebrated formula [20]. Let P_0 be the initial condition of equation (4.19) at time t_0, the solution of the equation (4.19) is given by

$$P(t) = \Phi'_{A_c}(t_o, t)P_0\Phi_{A_c}(t_0, t) - \int_{t_0}^{t} \Phi'_{A_c}(\tau, t)H(\tau)'H(\tau)\Phi'_{A_c}(\tau, t)d\tau. \qquad (4.21)$$

Without loss of generality, we assume $t_0 = 0$, then $P(T)$ is given by

$$P(T) = \Phi'_{A_c}(T, 0)^{-1}P_0\Phi_{A_c}(T, 0)^{-1} - \int_0^T \Phi'_{A_c}(\tau, T)H(\tau)'H(\tau)\Phi'_{A_c}(\tau, T)d\tau. \qquad (4.22)$$

Since we are looking for the periodic solutions of equations (4.19)-(4.20), the periodic generator P_0 must satisfy the following algebraic Lyapunov equation

$$P_0 = \Phi'_{A_c}(T, 0)P_0\Phi_{A_c}(T, 0) + W(T) \qquad (4.23)$$

where the $W(T)$ is given by

$$W(T) = W_c(T) - \Phi'_{A_c}(T, 0)Q_{tmp}(T)\Phi_{A_c}(T, 0). \qquad (4.24)$$

Proposition 4.1 *[21] The Lyapunov equations (4.19)-(4.20) admit a unique positive periodic solution iff the algebraic Lyapunov equation (4.23) admits a positive definite solution P_0.* $\qquad\qquad\Box$

Proposition 4.2 *Suppose that the $A_c(t)$ is asymptotically stable. Then the necessary condition that algebraic Lyapunov equation (4.23) admits a positive definite solution is that $W(T)$ defined in (4.24) is positive definite.* $\qquad\qquad\Box$

5   Proofs of Theorem 3.1-3.3

Now, we are ready to prove the main results of this paper. First we prove the Theorem 3.2, then we prove the Theorem 3.3. Finally, we consider the Theorem 3.1.

5.1 Theorem 3.2

Recall the hybrid system defined in (3.13)-(3.14),

$$
\begin{aligned}
\dot{x}(t) &= [A - B_1 B_1' P(t)]x(t), \\
&= A_c(t), \quad t \neq iT, \tag{5.1} \\
x(t^+) &= [I - \gamma^{-2} B_2 B_2' P(t)]x(t), \\
&= F_c(t), \quad t = iT. \tag{5.2}
\end{aligned}
$$

Denote the state transition matrix of the hybrid system (5.1) -(5.2) by $\Phi_c(t, \tau)$. It has the following properties,

$$
\begin{aligned}
\frac{\partial \Phi_c(t, \tau)}{\partial t} &= A_c(t)\Phi_c(t, \tau), \tag{5.3} \\
\Phi_c(\tau^+, \tau) &= F_c(\tau). \tag{5.4}
\end{aligned}
$$

Theorem 5.1 *Let $P(t)$ in equation(4.10) be substituted into equation (5.1) and (5.2). Also let $\lambda_i, i = 1, 2, \cdots \cdots n$ be n eigenvalues of $\Phi(T, 0)$ used to form Λ_1 in equation(4.6), then $\Phi_c(t, 0)$ is given by*

$$
\Phi_c(t, 0) = Y(t)Y(0)^{-1} \tag{5.5}
$$

and the eigenvalues of $\Phi_c(T, 0)$ are $\lambda_i, i = 1, 2, \cdots \cdots n$.

Proof: *Using equation (4.9), we have*

$$
\begin{aligned}
\dot{\Phi}_c(t, 0) &= \dot{Y}(t)Y(0)^{-1} \\
&= [AY(t) - B_1 B_1' X(t)]Y(0)^{-1} \\
&= [A - B_1 B_1' P(t)]\Phi_c(t, 0) \\
&= A_c(t)\Phi_c(t, 0). \tag{5.6}
\end{aligned}
$$

$$
\begin{aligned}
\Phi_c(0^+, 0) &= Y(0^+)Y(0)^{-1} \\
&= [Y(0) - \gamma^{-2} B_2 B_2' X(0)]Y(0)^{-1} \\
&= [I - \gamma^{-2} B_2 B_2' P(0)] \\
&= F_c(0). \tag{5.7}
\end{aligned}
$$

From equation (4.11),

$$
\begin{aligned}
Y(T) &= Y(0)\Lambda_1 \\
\Phi_c(T, 0) &= Y(T)Y(0)^{-1} = Y(0)\Lambda_1 Y(0)^{-1}. \tag{5.8}
\end{aligned}
$$

Thus, eigenvalues of $\Phi_c(T, 0)$ are $\lambda_i, i = 1, 2, \cdots \cdots n$. $\qquad\square$

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The following lemmas relate the positive definite periodic solution of the coupled Riccati equations and the structure properties of systems (controllability and observability).

Lemma 5.1 *[16] Assume (A, B_1) is controllable. A necessary and sufficient condition for the existence of a positive definite periodic solution $P(t)$ to equations (3.1) - (3.2) such that the hybrid system (5.1)-(5.2) is asymptotically stable is that no eigenvalue of $\Phi(T, 0)$ lies on the unit circle.* □

Lemma 5.2 *[16] Assume that no eigenvalue of $\Phi(T, 0)$ lies on the unit circle and (A, B_1) is controllable. Then there exists a unique positive definite periodic solution $P(t)$ to equations (3.1)-(3.2) if (C_1, A) is observable.* □

Proof of Theorem 3.2: It follows immediately from the Theorem 5.1 and Lemmas 5.1-5.2.

5.2   Theorem 3.3

(Sufficient condition):
Consider the following Lyapunov function candidate,

$$V(t) = x(t)'P(t)x(t). \tag{5.9}$$

Since $P(t) = P(t)' > 0$, then $V(t)$ is positive definite. $P(t)$ is a periodic function with jumps, so is the function $V(t)$. Hence we need to show that $V(t)$ is monotone decreasing, i,e.,

$$V(t_1) - V(t_2) < 0, \quad t_1 < t_2, \ \forall t_1, t_2. \tag{5.10}$$

We consider two cases:

(1) $t \neq iT$, differentiate $V(t)$ along the trajectory of system (3.12),

$$
\begin{aligned}
\dot{V}(t) &= \dot{x}(t)'P(t)x(t) + x(t)'\dot{P}(t)x(t) + x(t)'P(t)\dot{x}(t) \\
&= x(t)'(\dot{P}(t) + A_c(t)'P(t) + P(t)A_c(t))x(t) \\
&= -x(t)'H(t)'H(t)x(t) \\
&< 0.
\end{aligned}
\tag{5.11}
$$

Hence inequality holds for all $t_1, t_2 \neq iT$.

(2) $t = iT, i = 0, 1, 2, \cdots\cdots$. In the rest of proof, we use i to replace iT for the sake of brevity. Since $\dot{V}(t) = -x(t)'H(t)'H(t)x(t) < 0$,

$$
\begin{aligned}
V(i) &= V((i-1)^+) - \int_{i-1}^{i} x(\tau)'H(\tau)'H(\tau)x(\tau)d\tau \\
V(i^+) &= x(i)'P(i^+)x(i) \\
&= x(i)'P(i)x(i) + x(i)'Q_{tmp}(i)x(i) \\
&= V(i) + x(i)'Q_{tmp}(i)x(i).
\end{aligned}
\tag{5.12}
$$

Hence,

$$V(i^+) - V((i-1)^+) = x(i)'Q_{tmp}(i)x(i) - \int_{i-1}^{i} x(\tau)'H(\tau)'H(\tau)x(\tau)d\tau. \quad (5.13)$$

¿From equation (3.12), we can explicitly solve $x(t)$ starting at arbitrary $x(i-1)$,

$$x(t) = \Phi_{A_c}(t, i-1)x(i-1), \ \forall t \in [(i-1), i]. \quad (5.14)$$

Replace $x(t)$ in equation (5.13) by (5.14),

$$\begin{aligned}
&V(i^+) - V((i-1)^+) \\
&= x(i-1)'[\Phi'_{A_c}(i, i-1)Q_{tmp}(i)\Phi_{A_c}(i, i-1) \\
&\quad - \int_{i-1}^{i} \Phi'_{A_c}(\tau, i-1)H(\tau)'H(\tau)\Phi_{A_c}(\tau, i-1)d\tau]x(i-1). \quad (5.15)
\end{aligned}$$

Since $A_c(t), H(t)'H(t)$ and $Q_{tmp}(i)$ are periodic, $\Phi_{A_c}(i, i-1) = \Phi_{A_c}(T, 0)$, and $Q_{tmp}(T) = Q_{tmp}(T)$ for all $i = 0, 1, 2, \cdots\cdots$. The equation (5.15) can be further simplified as

$$\begin{aligned}
V(i^+) - V((i-1)^+) &= x(i-1)'[\Phi'_{A_c}(T, 0)Q_{tmp}(T)\Phi_{A_c}(T, 0) \\
&\quad - \int_0^T \Phi'_{A_c}(\tau, 0)H(\tau)'H(\tau)\Phi_{A_c}(\tau, 0)d\tau]x(i-1) \\
&< 0, \ by \ inequality \ (3.15). \quad (5.16)
\end{aligned}$$

(Necessary condition):
Since the closed-loop system $A_c(t)$ is asymptotically stable and $P(t)$ is positive definite, the inequality (3.15) follows immediately from Propositions 5.1 and 5.2. □

5.3   Theorem 3.1

We use the standard completion of squares method while accounting for the possible jumps. Differentiating $x'(t)P(t)x(t)$ along the trajectory of system (2.5), we obtain

$$\begin{aligned}
\frac{d}{dt}x'Px &= \dot{x}'(t)P(t)x(t) + x'(t)\dot{P}(t)x(t) + x'(t)P(t)\dot{x}(t) \\
&= x'(t)[A'P(t) + \dot{P}(t) + P(t)A]x(t) + 2 < u, B_1'Px > . \quad (5.17)
\end{aligned}$$

Replace term $[A'P(t) + \dot{P}(t) + P(t)A]$ by equation (3.1), and use assumption ii),

$$\begin{aligned}
\frac{d}{dt}x'Px &= -x'(t)C_1'C_1x(t) + x'(t)P(t)B_1B_1'P(t)x(t) + 2 < u, B_1'Px > \\
&= -||z||^2 + ||u + B_1'Px||^2. \quad (5.18)
\end{aligned}$$

Integrating equation (5.18) from $[iT, (i+1)T]$ for some i, the right-hand side (RHS) of equation (5.18) becomes

$$RHS = -||z||^2_{[iT,(i+1)T]} + ||u + B_1'Px||^2_{[iT,(i+1)T]}. \quad (5.19)$$

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where the left-hand side (LHS) of equation (5.17) becomes

$$
\begin{aligned}
LHS \;=\;& x'(iT^+)P(iT^+)x(iT^+) - x'(iT)P(iT)x(iT) \\
=\;& [x(iT) + B_2 v_0(i)]'P(iT^+)[x(iT) + B_2 v_0(i)] \\
& -x'(iT)P(iT)x(iT) \\
=\;& v_0'(i)B_2'P(iT^+)B_2 v_0(i) + 2 < v_0(i), B_2'P(iT^+)x(iT) > \\
& +x(iT)[P(iT^+) - P(iT)]x(iT) \\
=\;& -\gamma^2\|v_0(i)\|^2 + \|(\gamma^2 I + B_2 P(iT^+)B_2)^{1/2} \\
& \times [v_0(i) + (\gamma^2 I + B_2'P(iT^+)B_2)^{-1}B_2'P(iT^+)x(iT)]\|^2. \quad (5.20)
\end{aligned}
$$

Since the closed-loop system is stable, hence $x(t) \to 0$ as $t \to \infty$. Assume the initial condition $x(0) = 0$. Integrate above equation from 0 to ∞, we obtain the following equation,

$$
\begin{aligned}
& -\gamma^2\|v_0\|_{l_2}^2 + \sum_{i=0}^{\infty} \|(\gamma^2 I + B_2 P(iT^+)B_2)^{1/2} \\
& \times [v_0(i) + (\gamma^2 I + B_2'P(iT^+)B_2)^{-1}B_2'P(iT^+)x(iT)]\|^2 \\
=\;& -\|z\|_{L_2}^2 + \|u + B_1'Px\|_{L_2}^2. \quad (5.21)
\end{aligned}
$$

Taking a feedback controller $u(t) = -B_1'P(t)x(t)$ and since the summation term is always nonnegative. Thus we have the following inequality

$$
\|z\|_{L_2}^2 < \gamma^2\|v_0\|_{l_2}^2. \quad (5.22)
$$

It is easy to see from (5.21) that the worst case impulsive disturbances is

$$
v_0(i) = -(\gamma^2 I + B_2'P(kT^+)B_2)^{-1}B_2'P(iT^+)x(iT). \quad (5.23)
$$

$\square$

6 Conclusion

In this paper, we have given a complete state-space solution to the (sub)optimal control problem of a class of linear systems suject to impulsive disturbances. The state feedback controller can be computed in terms of the unique positive definite periodic solution of a coupled Riccati equations. The procedure to compute such a feedback controller is outlined as follows:

1). Given $\gamma > 0$, check if the matrix $\Phi(T, 0)$ has no eigenvalues on the unit circle. If not, increase γ until the condition is satisfied.

2). Compute the periodic solution $P(t)$ according to equations (4.3), (4.7) and (4.10).

3). Check if the inequality (3.15) is satisfied. If not, go back to step 1) and increase γ until this condition is satisfied.

4). Compute the feedback controller $u(t)$ according to equation (3.3).

The results obtained in this paper can be extended into the output feedback case, or more interestingly, into the nonlinear impact control problems [2, 16].

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